Indian Statistical Institute, Bangalore

B. Math.

Third Year, First Semester

Analysis on Graphs

Back paper Examination — Date : December 29, 2023

Maximum marks: 100 — Time: 3 hours

Instructor: B V Rajarama Bhat

In the following G is a simple graph (undirected and without loops), with vertex set $V(G) = \{1, 2, \ldots, n\}$ and edge set $E(G) = \{e_1, \ldots, e_m\}$.

(1) Let G be a connected graph with adjacency matrix A. Suppose G has diameter d. Show that A has at least $d + 1$ distinct eigenvalues. [21]

(2) Show that the number of spanning trees of a graph G of n vertices with Laplacian L is equal to

$$\frac{1}{n}\lambda_1.\lambda_2.\ldots.\lambda_{n-1},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of L arranged in decreasing order. (You may carefully state the matrix-tree theorem and use it). [21]

(3) Show that the algebraic connectivity is monotone, that is, if $ac(G)$ denotes the second smallest eigenvalue of the Laplacian of G, and G is edge-disjoint union of two graphs H_1, H_2 on same vertex set, then

$$ac(G) \geq ac(H_1) + ac(H_2) \geq ac(H_1).$$

[21]

(4) (Rook's graph) Fix a natural number $m \geq 2$. Let R_m be a graph with vertex set equal to $\{(i, j) : 1 \leq i, j \leq m\}$. Further, distinct vertices (i_1, j_1) and (i_2, j_2) form an edge if and only if either $i_1 = i_2$ or $j_1 = j_2$. Show that R_m is a strongly regular graph. Compute the parameters of strong regularity. Use this to compute the spectrum of R_m. Recall that the spectrum of a strongly regular graph with parameters (n, k, a, c) are k, λ_+, λ_- with multiplicities $1, m_+, m_-$ respectively, where

$$\lambda_\pm = \frac{1}{2}(a - c \pm \sqrt{\Delta}),$$

and

$$m_\pm = \frac{1}{2}[(n-1) \pm \frac{(n-1)(c-a) - 2k}{\sqrt{\Delta}}],$$

with $\Delta = (a - c)^2 + 4(k - c)$.

[21]

(5) Let $N = \{0, 1, 2, 3, 4, 5\}$ considered as a group under addition modulo 6. Let H be the Cayley graph of N with generating set $S = \{2, 3, 4\}$. Write down adjacency, Laplacian and distance matrices of H. [21]